From relations (12) and (14), the displacement is of the form

$$\vartheta = \sum_{n=1}^{\infty} \frac{-2K_nPa^2J_2(K_na)J_1(K_nr)e^{-mz}e^{-\Omega t}}{(\mu - \mu'\Omega)m\{(K_n{}^2a^2 - 1)J_1{}^2(K_na) + a^2K_n{}^2J_1{}'^2(K_na)\}}$$

$$= \frac{-2Pa^2e^{-\Omega t}}{(\mu - \mu'\Omega)} \sum_{n=1}^{\infty} \frac{K_nJ_2(K_na)J_1(K_nr)e^{-mz}}{m[(K_n{}^2a^2 - 1)J_1{}^2(K_na) + a^2K_n{}^2J_1{}'^2(K_na)]} \tag{15}$$

From (15) and (4), one obtains

$$\tau_{\theta z} = [\mu + \mu'(\partial/\partial t)](\partial\vartheta/\partial z)$$

$$= 2e^{-\Omega t}\,\rho a^2 \cdot \sum_{n=1}^{\infty} \frac{K_nJ_2(K_na)J_1(K_nr)e^{-mz}}{\{(K_n{}^2a^2 - 1)J_1{}^2(k_na) + a^2K_n{}^2J_1{}'^2(K_na)\}} \tag{16}$$

$$\tau_{\theta r} = \left(\mu + \mu'\frac{\partial}{\partial t}\right)\left(\frac{\partial\vartheta}{\partial r} - \frac{\vartheta}{r}\right) =$$

$$-2pa^2e^{-\Omega t}\left\{\sum_{n=1}^{\infty} \frac{K_n{}^2J_2(K_na)J_0(K_nr)e^{-mz}}{m[(K_n{}^2a^2 - 1)J_1{}^2(K_na) + a^2K_n{}^2J_1{}'^2(K_na)]} - \frac{2}{r}\sum_{n=1}^{\infty} \frac{K_nJ_2(K_na)J_1(K_nr)}{m[(K_n{}^2a^2 - 1)J_1{}^2(K_na) + a^2K_n{}^2J_1{}'^2(K_na)]}\right\} \tag{17}$$

On the plane boundary $Z = 0$, the displacement, from (15), is

$$[\vartheta]_{z=0} = \frac{-2pa^2e^{-\Omega t}}{(\mu - \mu'\Omega)} \sum_{n=1}^{\infty} \frac{K_nJ_2(K_na)J_1(K_nr)}{m[(K_n{}^2a^2 - 1)J_1{}^2(K_na) + a^2K_n{}^2J_1{}'^2(K_na)]} \tag{18}$$

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Method of Evaluating Script *F* for Radiant Exchange within an Enclosure

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A method of evaluating script *F* for radiant exchange within an enclosure by a single matrix inversion is developed. The method lends itself well to problems handled analytically or to problems solved by the use of a digital computer.

Nomenclature

- a_{jk} = element of the inverse matrix $[\beta]^{-1}$ (jth row, kth column), dimensionless
- A_j = area of the jth surface, ft²
- F_{jk} = shape factor from jth surface to kth surface
- J_j = radiosity of the jth surface
- $q_{j,k}$ = net thermal radiation exchange between the jth surface and the kth surface, Btu/hr
- ϵ_j = emissivity of the jth surface
- ρ_j = reflectivity of the jth surface
- σ = Stefan-Boltzmann constant

Received March 4, 1963.

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THE thermal interchange between diffuse surfaces in an enclosure usually is evaluated by one of the two methods, the network or the script *F* method. Both methods are established well and are presented in Refs. 1 and 2. It is shown readily that the basic equations of the net radiation concept as formulated by Poljak[3] leads either to the Oppenheim network or to the script *F* concept employed by Hottel.

Coupling of the radiation network which is described by n-simultaneous linear equations (linear in terms of σT^4 and n is the number of surfaces) with the temperature-time dependent equations of conduction and convection is somewhat unwieldy since, at each interval of time, the n-simultaneous equations must be solved. It is simpler to employ the concept of script *F* and linearize the thermal radiation exchange.

Analysis

The basis of the script *F* method is that the net exchange between surfaces A_j and A_k in an enclosure must be of the form $\sigma(T_j{}^4 - T_k{}^4)$ multiplied by some factor $\mathfrak{F}_{jk}$ (called script *F*) which depends upon the geometry and the emissivity of the surfaces. Thus the net exchange between two surfaces identified by the subscripts j and k can be expressed in the form

$$q_{j,k} = A_j\mathfrak{F}_{jk}\,(\sigma T_j{}^4 - \sigma T_k{}^4) \tag{1}$$

Script F, $\mathfrak{F}_{jk}$, is found from the n-simultaneous equations that describe the Oppenheim network. It is shown readily that the set of radiosity equations are

$$J_j - \epsilon_j\,\sigma T_j{}^4 = \rho_j \sum_{k=1}^{n} F_{jk}\,J_k \qquad j = 1, 2, 3, \ldots n \tag{2}$$

If all the terminals ($\sigma T_k{}^4$ is a potential at the terminal k) are grounded except the jth, which is fixed at unity potential, the net flow, $q_{j,k}$, into the grounded kth terminal is

$$(J_k\,\epsilon_k\,A_k)/\rho_k \tag{3}$$

This relationship can be verified easily by examining the network.

Since $\sigma T_j{}^4 = 1.0$ and $\sigma T_k{}^4 = 0$, $k = 1, 2, \ldots 3, \ldots n$, $k \neq i$, it is seen from (1) and (3) that

$$q_{j,k}{}' = A_j \qquad \mathfrak{F}_{jk} = J_k\,\epsilon_k\,A_k/\rho_k$$

or

$$\mathfrak{F}_{jk} = J_k\,\epsilon_k\,A_k/\rho_k\,A_j \tag{4}$$

Radiosity, J_k, is found from the set of Eqs. (2) with the terminals grounded except the jth, which is set at unity potential. In matrix form, this can be expressed as

$$[\beta]\begin{Bmatrix} J_1 \\ \cdot \\ \cdot \\ \cdot \\ J_k \\ \cdot \\ \cdot \\ \cdot \\ J_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \epsilon_j \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \tag{5}$$

or

$$\begin{Bmatrix} J_1 \\ J_k \\ J_n \end{Bmatrix} = [\beta]^{-1}\begin{Bmatrix} 0 \\ \epsilon_j \\ 0 \end{Bmatrix} \tag{6}$$

where

$$[\beta] = \begin{bmatrix} 1 - \rho_1 F_{11} & -\rho_1 F_{12} \ldots -\rho_1 F_{1n} \\ -\rho_2 F_{21} & 1 - \rho_2 F_{22} \ldots -\rho_2 F_{2n} \\ \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot \quad\quad \cdot \\ -\rho_n F_{n1} & -\rho_n F_{n2} \ldots 1 - \rho_n F_{nn} \end{bmatrix}$$

Solution of Eq. (6) reveals that

$$J_k = a_{kj}\, \epsilon_j \tag{7}$$

where a_{kj} (the kth row, jth column) is the element of the inverse matrix $[\beta]^{-1}$. Substitution of (7) into (4) yields

$$\mathfrak{F}_{jk} = a_{kj}\, \epsilon_k\, A_k\, \epsilon_j / \rho_k\, A_j \tag{8}$$

By the direct expansion of $[\beta]^{-1}$, it is established readily that

$$a_{kj}\, A_k/\rho_k = a_{jk}\, A_j/\rho_j$$

thus

$$\mathfrak{F}_{jk} = a_{jk}\, (\epsilon_j\, \epsilon_k/\rho_j)$$

This technique readily yields script F by a single matrix inversion. The method lends itself well to problems handled analytically with n small or for problems solved by the use of a digital computer with n large.

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Motion of an Asymmetric Spinning Body with Internal Dissipation

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Introduction

The moment-free motion of a spinning body with internal energy dissipation is important in satellite wobble-damping problems. The important relationship for such problems is that between the kinetic energy of rotation T of the body and the half-cone angle θ of precession. For a symmetric body this relationship is well known.[1, 2]

The mathematical solution for the moment-free motion of an asymmetric spinning body results in elliptic functions and may be found, for example, in Ref. 3. The purpose of this note is to put the classical results for the attitude drift of an asymmetric body into the form most convenient for treating the problems of internal energy dissipation and to show that, when formulated properly, the results take on a form very similar to those for the symmetric case.

Theory

For a nearly rigid symmetric body with principal axes 1, 2, 3 and moments of inertia I, I, J (Fig. 1) the attitude angle θ between the spin axis 3 and the constant angular momentum vector h can be expressed in terms of the kinetic energy T by starting with the momentum and energy equations

$$h^2 = I^2(\omega_1^2 + \omega_2^2) + J^2\omega_3^2 \tag{1}$$

$$2T = I(\omega_1^2 + \omega_2^2) + J\omega_3^2 \tag{2}$$

and noting that $\cos\theta = J\omega_3/h$.

The energy and momentum associated with the internal motion is neglected. The desired relationship is then

$$\cos^2\theta = [(h^2 - 2TI)J/(J - I)h^2] \tag{3}$$

which upon differentiation results in the equation

$$(2 \sin\theta \cos\theta)\dot{\theta} = -[2\dot{T}IJ/(I - J)h^2] \tag{4}$$

Since $\dot{T}$ for energy dissipation is negative, $\dot{\theta}$ must be negative for $J > I$ and positive for $J < I$.

To solve for θ as a function of time the small energy dissipation rate $\dot{T}$ is obtained from an examination of the internal dissipative mechanisms within the body. The technique used is outlined for particular cases in References 1, 2, 4 and 5. For the purposes of this discussion $\dot{T}$ may be assumed known.

For the asymmetric body with moments of inertia A, B, C, about principal axes 1, 2, 3, the solutions for the angular velocities are in terms of elliptic functions. Assume $A > B > C$ and $h^2 < 2TB$. This corresponds to the initial conditions for a long, thin body spinning about its long axis Then the angular velocities are given as:[3]

$$\begin{aligned} \omega_1 &= \left[\frac{h^2 - 2TC}{A(A - C)}\right]^{1/2} cn\; p(t - t_0) \\ \omega_2 &= \left[\frac{h^2 - 2TC}{B(B - C)}\right]^{1/2} sn\; p(t - t_0) \\ \omega_3 &= -\left[\frac{2TA - h^2}{C(A - C)}\right]^{1/2} dn\; p(t - t_0) \\ p &= \left[\frac{(B - C)(2TA - h^2)}{ABC}\right]^{1/2} \end{aligned} \tag{5}$$

and the modulus k of the elliptic functions is

$$k = \left[\frac{(A - B)}{(B - C)}\frac{h^2 - 2CT}{2AT - h^2}\right]^{1/2}$$

The properties of the elliptic functions indicate that ω_1 and ω_2 are oscillatory with zero mean value, while ω_3 fluctuates slightly but is never zero. The motion is then a spin about axis 3 with a slight wobble.

On eliminating ω_1 from the momentum and energy equations, one can write

$$\left(\frac{C\omega_3}{h}\right)^2 = \cos^2\theta = \frac{(2TA - h^2)C}{(A - C)h^2} \times \left[1 - \frac{(A - B)(h^2 - 2TC)}{(B - C)(2TA - h^2)}\, sn^2\, p(t - t_0)\right] \tag{6}$$

Since $sn^2\, p(t - t_0)$ oscillates between one and zero, Eq. (6) indicates that the nutation or cone angle θ, which is not oscillatory for a symmetric body, fluctuates with frequency $2p$ for an asymmetric body. If one chooses as new variables the maxi-

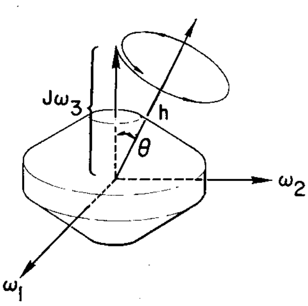

Fig. 1 Motion of free spinning body

Received February 18, 1963.

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